## Group actions, revisited

Recall: If a group $G$ acts on a nonempty set $A$, then for every $g \in G$, the function

$$\sigma_g : A \to A \quad \text{defined } \sigma_g(a) = g \cdot a$$

is a bijection, and the function

$$\varphi : G \to S_A \quad \text{defined } \varphi(g) = \sigma_g$$

is a homomorphism. The kernel of $\varphi$ is called the kernel of the action, and the action is faithful if the kernel is the identity.

Note: The kernel of the action $K$ is a normal subgroup and we can give $G/K$ an action on $A$ as follows:

$$gK \cdot a = g \cdot a.$$

Check this is well-defined: If $gK = hK$ then $g = hk$ for some $k \in K$.

$$\Longrightarrow \quad g \cdot a = hk \cdot a = h \cdot (\underbrace{k \cdot a}_{\text{acts as identity}}) = h \cdot a$$

Then $G/K \cong \operatorname{im}\varphi \leq S_A$, so it has trivial kernel, so it's faithful!

## Group actions from maps to $S_A$

We can also get group actions from morphisms to $S_A$. i.e. if $A$ is a set and $G$ a group s.t.

$\varphi : G \to S_A$ is a homomorphism, define the group action of $G$ on $A$ as follows:

$$\text{For } g \in G,\ a \in A, \quad g \cdot a = \underbrace{\varphi(g)}_{\substack{\text{This is}\\ \text{a bijection}\\ A \to A}}(a)$$

This is in fact an action (axioms are easy to check), and all actions of $G$ on $A$ arise in this way. i.e.

**Prop:** There is a bijection between the actions of $G$ on $A$ and the homomorphisms $G \to S_A$.

**Def:** If $G$ is a group, a <u>permutation representation</u> of $G$ is any homomorphism $G \to S_A$, for some nonempty set $A$, thus giving an action of $G$ on $A$.

## Orbits and stabilizers

Let $G$ be a group acting on a set $A$. Recall from a HW problem that we can define an equivalence relation on $A$ as follows:

$$a \sim b \iff a = g \cdot b, \text{ some } g \in G.$$

We already showed this is an equivalence relation, and the equivalence classes are called <u>orbits</u>. For $a \in A$, the equivalence class containing $a$ is called the orbit of $a$.

Note that the stabilizer of $a$, $G_a := \{g \in G \mid g \cdot a = a\}$ gives us no additional elements of the orbit of $a$, and in fact, each coset of $G_a$ gives us an additional element of the orbit of $a$. That is,

**Prop:** For $a \in A$, the number of elements in the orbit of $a$ is $|G : G_a|$.

**Pf:** We show that there is a bijection between the cosets of $G_a$ and the elts of the orbit of $a$. Call the orbit $\mathcal{O}_a$.

We define a map $\mathcal{O}_a \to$ cosets of $G_a$ by

$$b = g \cdot a \longmapsto g G_a.$$

This is well-defined: If $b = g \cdot a = h \cdot a$, then $h^{-1} g \cdot a = a \Rightarrow h^{-1} g \in G_a$. Thus, $(h^{-1} g) G_a = 1 G_a \Rightarrow g G_a = h G_a$.

This is clearly surjective, and it's injective since if $g G_a = h G_a$, then $h^{-1} g \in G_a \Rightarrow h^{-1} g \cdot a = a \Rightarrow g \cdot a = h \cdot a$. $\square$

**Def:** The action is <u>transitive</u> if there's only one orbit. i.e. if $\forall\ a, b \in A\ \ \exists\, g \in G$ s.t. $a = g \cdot b$.

**Ex:** $S_n$ always acts transitively on $\{1, \dots, n\}$: if $a, b \in \{1, \dots, n\}$, $(a\, b) \cdot a = b$.